Nonhomogeneous Problems (Sects. 3.4 and 8.2, 8.3).

Consider the following problems of Sect. 3.4

\# 3.4.9.

$$\begin{cases} u_t = K u_{xx} + q(x,t) \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

\# 3.4.11

$$\begin{cases} u_t = K u_{xx} + g(x) \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

\# 3.4.12

$$\begin{cases} u_t = K u_{xx} + e^{-t} + e^{-2t} \cos \frac{3\pi}{L} x \\ u_x(0,t) = 0, \quad u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

\# 3.4.13

$$\begin{cases} u_t = K u_{xx} \\ u(0,t) = A(t), \quad u(L,t) = 0 \\ u(x,0) = g(x) \end{cases}$$

Ask: why MSVs does not work?

Most general problem for Heat cond. (1-D.)

Dirichlet B.C.

$$\begin{cases} U_t = kU_{xx} + Q(x,t), & 0 < x < L, \quad t > 0 \\ U(x,0) = f(x) \\ U(0,t) = A(t) \\ U(L,t) = B(t) \end{cases}$$

CHAPTER 8

Nonhomogeneous Problems

8.2 and 8.3

Steady Nonhomog. equation and BC's.

$$\text{IBVP}\begin{cases} u_t = K u_{xx} + Q(x) \\ u(0,t) = A, \quad u(L,t) = B \\ u(x,0) = f(x) \end{cases}$$

For this problem, there is an equilibrium solution : $\hat{u}(x)$

that satisfies

$$\begin{cases} K\hat{u}_{xx} = -Q(x) \\ \hat{u}(0) = A, \quad \hat{u}(L) = B \end{cases}$$

Defining $\boxed{v(x) \equiv u(x,t) - \hat{u}(x)}$

Then, $v(x)$ satisfies

$$v_t = u_t - (\hat{u})_t^{\nearrow 0} = K u_{xx} + Q(x) = K u_{xx} - K\hat{u}_{xx}$$

$$= K(u_{xx} - \hat{u}_{xx}) = K(u - \hat{u})_{xx}$$

$$= K\, v_{xx}(x,t)$$

or $\boxed{v_t = K v_{xx}}$

Also,

$$v(0,t) = u(0,t) - \hat{u}(0) = A - A = 0$$

$$v(L,t) = u(L,t) - \hat{u}(L) = B - B = 0.$$

$$v(x,0) = u(x,0) - \hat{u}(x) = f(x) - \hat{u}(x).$$

Therefore, the new function $v(x)$ satisfies:

$$\begin{cases} v_t = k v_{xx} \\ v(0,t) = 0, \quad v(L,t) = 0 \\ v(x,0) = \boxed{f(x) - \hat{u}_x} \text{ difference.} \end{cases}$$

Then, we can use sep. of variables to solve it. (ASK for Neumann problem)

Time-dependent nonhomogeneous terms.

$$\begin{cases} u_t = k u_{xx} + Q(x,t) \\ u(0,t) = A(t), \quad u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

Not an equillibrium.

Define $\boxed{r(x,t) = A(t) + \frac{x}{L}\left(B(t) - A(t)\right)}$

and $v(x,t) = u(x,t) - r(x,t)$

Then

$$v_t = u_t - r_t = Ku_{xx} + Q(x,t) - r_t = Ku_{xx} - K\overset{0}{r_{xx}} + Q(x,t) - r_t$$

$$= K v_{xx} + Q(x,t) - r_t$$

or

$$\boxed{v_t = K v_{xx} + Q(x,t) - r(t)}$$

$$v(0,t) = u(0,t) - r(0,t) = A(t) - A(t) = 0$$

$$v(L,t) = u(L,t) - r(L,t) = B(t) - B(t) = 0$$

$$v(x,0) = u(x,0) - r(x,0) =$$

$$= f(x) - \left[A(0) + \frac{x}{L}\left(B(0) - A(0)\right)\right]$$

Calling $\bar{Q}(x,t) \equiv Q(x,t) - r(t)$

$$\bar{f}(x) = f(x) - \left[A(0) + \frac{x}{L}(B(0) - A(0))\right]$$

The IBVP for $v(x,t)$ is given by

$$\begin{cases} v_t = K v_{xx} + \bar{Q}(x,t) & (3.1) \\ v(0,t) = 0, \quad v(L,t) = 0 & (3.2) \\ v(x,0) = \bar{f}(x) & (3.3) \end{cases}$$

Nonhomogeneous equ. with homogeneous B.C's.

8.3.7 Solve the IBVP

$$\begin{cases} u_t = u_{xx}, & 0<x<L,\ t>0 & (1) \\ u(0,t)=0 & & (2) \\ u(L,t)=t & & (3) \\ u(x,0)=0 & & (4) \end{cases}$$

Because B.C. depends on time there is not an equilibrium solution.

But, we still can introduce a change of dependent variables that will transform our original homogeneous equation into a nonhomogeneous one, but the B.C. will be homogeneous. In fact, defining

$$\boxed{r(x,t) \equiv A(t) + \frac{(B(t)-A(t))}{L}x = \frac{t}{L}x}$$

and $\boxed{v(x,t) \equiv u(x,t) - r(x,t).}$ $\Rightarrow u(x,t) = v(x,t) + r(x,t)$

$$\Rightarrow \quad 0 = u_t - u_{xx} = v_t + r_t - v_{xx} - r_{xx} =$$

$$= v_t + \frac{x}{L} - v_{xx} - 0$$

$$\Rightarrow \quad \boxed{v_t = v_{xx} - \frac{x}{L} = Q(x)}$$

The B.C's transform into homogeneous B.C's. for $v(x,t)$.

$$v(0,t) = u(0,t) - r(0,t) = 0 - 0 = 0$$

$$v(L,t) = u(L,t) - r(L,t) = t - t = 0$$

and the I.C. transform into

$$v(x,0) = u(x,0) + r(x,0) = 0 + 0 = 0.$$

Summarizing, the new IBVP for the new dependent variable $v(x,t)$ is given by

$$\begin{cases} v_t = v_{xx} - \dfrac{x}{L} & (5) \\ v(0,t) = 0, \quad v(L,t) = 0 & (6) \\ v(x,0) = 0 & (7) \end{cases}$$

this new problem <u>admits an equilibrium solution $v_E(x)$</u>

In fact,

$$\begin{cases} v_E''(x) = \dfrac{x}{L} \\ v_E(0) = 0, \quad v_E(L) = 0 \end{cases}$$

$$\Rightarrow v_E'(x) = \frac{x^2}{2L} + C_1 \Rightarrow v_E(x) = \frac{x^3}{6L} + C_1 x + C_2$$

Using the B.C's. $\quad 0 = v_E(0) = C_2 \Rightarrow C_2 = 0$

and $0 = v_E(L) = \dfrac{L^{\cancel{3}\,2}}{6\cancel{L}} + C_1 L \Rightarrow C_1 = -\dfrac{L}{6}$

$$\Rightarrow \boxed{v_E(x) = \frac{x^3}{6L} - \frac{L}{6}x}$$

Therefore, if we define

$$\boxed{\hat{v}(x,t) = v(x,t) - v_E(x)}, \quad \text{or} \quad v(x,t) = \hat{v}(x,t) + v_E(x)$$

The new dependent variable $\hat{v}(x,t)$ will satisfy

$$0 = v_t - v_{xx} + \frac{x}{L} = \hat{v}_t + 0 - \hat{v}_{xx} - v_E''(x) + \frac{x}{L}$$

$$= \hat{v}_t - \hat{v}_{xx} - \frac{x}{L} + \frac{x}{L}$$

$$\Rightarrow \begin{cases} \hat{v}_t = \hat{v}_{xx} \\ \hat{v}(0,t) = \overset{=0}{v(0,t)} - \overset{=0}{v_E(0)} = 0 \\ \hat{v}(L,t) = v(L,t) - v_E(L) = 0 - 0 = 0 \\ \hat{v}(x,0) = v(x,0) - v_E(x) = -v_E(x) = \frac{L}{6}x - \frac{x^3}{6L} \end{cases}$$

whose solution is given (after sep. of variables) by

$$\hat{v}(x,t) = \sum_{n=1}^{\infty} \hat{a}_n \sin \frac{n\pi}{L} x \, e^{-\left(\frac{n\pi}{L}\right)^2 t} \quad \text{where} \quad \boxed{\hat{a}_n \equiv \frac{2}{L} \int_0^L \left[\frac{L}{6}x - \frac{x^3}{6L}\right] \sin \frac{n\pi}{L} x \, dx}$$

$$\Rightarrow \boxed{v(x,t) = \hat{v}(x,t) + v_E(x) = \sum_{n=1}^{\infty} \hat{a}_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} + \frac{x^3}{6L} - \frac{L}{6}x}$$

and finally

$$\boxed{u(x,t) = v(x,t) + r(x,t) = \sum_{n=1}^{\infty} \hat{a}_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} + \frac{x^3}{6L} - \frac{L}{6}x + \frac{t}{L}x}$$

In a more general case, eigenfns expansion is suggested

First, the eigenfunctions are determined by considering the related homogeneous problem

$$\begin{cases} w_t = k w_{xx} + \bar{Q}(x,t) & (4.1) \\ w(0,t) = 0, \quad w(L,t) = 0 & (4.2) \\ w(x,0) = \bar{f}(x) & (4.3) \end{cases}$$

Sep. of variables leads to EVP:

$$\phi'' + \lambda \phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, \ldots$$

Second, the unknown function $w(x,t)$ is expanded in terms of these eigenfunctions.

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi}{L}x\right) \qquad (4.4)$$

with unknown coeffs $a_n(t)$ depending on t.

$w(x,t)$ defined by (4.4) satisfies B.C. and it also satisfies I.C. if

$$w(x,0) = \sum_{n=1}^{\infty} a_n(0) \sin\left(\frac{n\pi}{L}x\right) = \bar{f}(x)$$

Which is true if

$$a_n(0) = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

To determine $a_n(t)$, we assume that derivation term by term is allowed (True if $v(x,t)$ satisfies same homogeneous BC's as $\phi_n(t)$)

Subst. into (3.1).

$$\sum_{n=1}^{\infty}\left[a_n'(t)\,\phi_n - k\,a_n(t)\,\phi_n''(x)\right] = \bar{Q}(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)$$

but $\phi_n''(x) = -\lambda_n\phi_n(x)$

$$\Rightarrow \sum_{n=1}^{\infty}\left[a_n'(t)\,\phi_n + \lambda_n k\,a_n(t)\,\phi_n(x)\right] = \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)$$

orthogonality

$$\begin{cases} a_n'(t) + \lambda_n k\,a_n(t) = q_n(t), & n = 1, 2, \ldots \quad (8.1) \\ a_n(0) = \dfrac{\int_0^L f(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2} & \checkmark \quad (8.2) \end{cases}$$

Solve this IVP for 1^{st} order linear equation to obtain $\underline{\underline{a_n(t)}}$.

Finally,

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) \Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) + r(x,t).}$$

8.3.3

IVP:

$$c \rho u_t = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + q u + f(x,t) \qquad (1.1)$$

c, ρ, K_0, q functions of x

Satisfying: $u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = g(x)$

IBVP with homog. equ.

$$\begin{cases} c \rho v_t = \frac{\partial}{\partial x}\left(K_0 \frac{\partial v}{\partial x}\right) + q v \\ \text{Same BCs. and IC.} \end{cases}$$

MSV: $v(x,t) = h(t)\, \phi(x)$

$$c \rho\, h'(t)\, \phi(x) = \frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) h(t) + q\, h(t)\, \phi(x)$$

$\frac{1}{c(x)\rho(x)\, h\, \phi}$

$$\frac{c(x)\rho(x)\, h'(t)}{h(t)} = \frac{1}{\phi}\frac{d}{dx}\left(K_0(x) \frac{d\phi}{dx}\right) + q = -\lambda$$

$$\Rightarrow \quad \boxed{h'(t) + \lambda\, h(t) = 0} \qquad (1.2)$$

and Eigenvalue problem

$$\begin{cases} \frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + q\phi = -\lambda \overbrace{c(x)\rho(x)}^{=\sigma(x)} \phi \qquad (1.3) \\ \phi(0) = 0, \quad \phi(L) = 0 \end{cases}$$

Since EVP is of Sturm-Liouville Type there are a complete set of eigenfunctions $\{\phi_n(x)\}$ Corresp. to eigenvalues $\{\lambda_n\}$

and
$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x)$$

Subst. into (1.1). assuming that derivation term by term is possible.

$$c\rho \sum_{n=1}^{\infty} a_n'(t)\, \phi_n(x) = \sum_{n=1}^{\infty} a_n(t) \left[\frac{d}{dx}\left(K_0 \frac{d\phi_n}{dx}\right) + q\phi_n \right] + f(x,t)$$

using (1.3)
$\Rightarrow$
$$c\rho \sum_{n=1}^{\infty} a_n'(t)\, \phi_n(x) = \sum_{n=1}^{\infty} a_n(t) \left(-\lambda_n \underbrace{\sigma(x)}_{=c\rho} \phi_n(x)\right) + f(x,t)$$

or
$$\sum_{n=1}^{\infty} \left[a_n'(t) + \lambda_n a_n(t) \right] \phi_n(x) = \frac{1}{\underbrace{c(x)\rho(x)}_{=\sigma(x)}} f(x,t)$$

Use orthog. $\left[a_n'(t) + \lambda_n a_n(t)\right] \int_0^L \phi_n^2(x)\, \sigma(x)\, dx = \int_0^L f(x,t)\, \phi_n(x) \frac{\sigma(x)}{\sigma(x)}\, dx$

$$\Rightarrow \quad a_n'(t) + \lambda_n a_n(t) = \frac{\int_0^L f(x,t)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, \sigma(x)\, dx}$$

and from I.C.

$$g(x) = U(x,0) = \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x)$$

$$\Rightarrow \qquad a_n(0) = \frac{\int_0^L g(x)\phi_n(x)\sigma(x)\,dx}{\int_0^L \phi_n^2(x)\,\sigma(x)\,dx}.$$